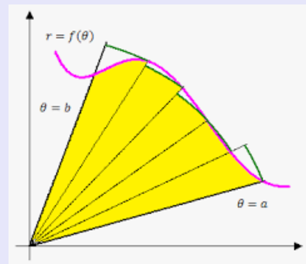
Calculus II

Lecture 20



Feb 19-8:47 AM

Class Qz 17 (open Notes)

Given $r = 8\sin\theta$

1) Convert to rectangular eqn.

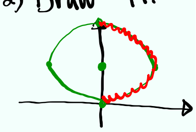
$r = 8\sin\theta$ $\quad x^2+y^2=8y$

$r^2 = 8r\sin\theta$ $\quad x^2+y^2-8y+16=16$

$x^2+(y-4)^2=4^2$

Circle Center $(0,4)$ Radius 4

2) Draw it.



3) Find the arc length using the arc length formula $L=\int_\alpha^\beta \sqrt{r^2+\left[\frac{dr}{d\theta}\right]^2}\,d\theta$ for the curve in the first quadrant.

$$L=\int_0^{\pi/2}\sqrt{(8\sin\theta)^2+(8\cos\theta)^2}\,d\theta$$

$$=\int_0^{\pi/2}\sqrt{64(\sin^2\theta+\cos^2\theta)}\,d\theta=\int_0^{\pi/2}8\cdot 1\,d\theta=8\theta\Big|_0^{\pi/2}$$

$$=8\left(\frac{\pi}{2}\right)=\boxed{4\pi}$$

$\dfrac{\text{Circumference}}{2}$ $\quad \dfrac{C}{2}=\dfrac{2\pi R}{2}$

$=\pi R$

$=\boxed{4\pi}$

Jul 15-6:59 AM

Given $a_n = \frac{(-1)^{n+1}}{5^n}$ $\quad \frac{1}{5}, \frac{-1}{25}, \frac{1}{125}, \frac{-1}{625}, \ldots$

1) Write out the first 3 terms. Alternating

$n=1$ $\quad a_1 = \frac{(-1)^2}{5^1} = \frac{1}{5}$ $\qquad n=2$ $\quad a_2 = \frac{(-1)^3}{5^2} = \frac{-1}{25}$ $\qquad n=3$ $\quad a_3 = \frac{(-1)^4}{5^3} = \frac{1}{125}$

2) Simplify $\frac{a_{n+1}}{a_n}$

$$\frac{\frac{(-1)^{n+2}}{5^{n+1}}}{\frac{(-1)^{n+1}}{5^n}} = \frac{(-1)^{n+2} \cdot 5^n}{(-1)^{n+1} \cdot 5^{n+1}} = \frac{(-1)^1}{5^1} = \frac{-1}{5}$$

$\frac{a_{n+1}}{a_n} < 1$

$|a_{n+1}| < |a_n| \rightarrow$ Decreasing Sequence

3) what can we conclude about increasing, or decreasing of $\{|a_n|\}_{n=1}^{\infty}$

$|a_n| = \frac{1}{5^n}$ $\qquad \frac{a_{n+1}}{a_n} = \frac{\frac{1}{5^{n+1}}}{\frac{1}{5^n}} = \frac{5^n}{5^{n+1}} = \frac{1}{5} < 1$

Decreasing

Jul 16-8:22 AM

Given $\left\{ \frac{1}{(n-1)!} \right\}_{n=1}^{\infty}$

1) Write out the first 4 terms.

$n=1$ $\quad a_1 = \frac{1}{0!} = \frac{1}{1} = 1$ $\qquad n=2$ $\quad a_2 = \frac{1}{1!} = 1$ $\qquad n=3$ $\quad a_3 = \frac{1}{2!} = \frac{1}{2}$ $\qquad n=4$ $\quad a_4 = \frac{1}{3!} = \frac{1}{6}$

2) Find $\frac{a_{n+1}}{a_n}$, Is the sequence increasing or decreasing?

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1-1)!}}{\frac{1}{(n-1)!}} = \frac{(n-1)!}{n!} = \frac{\cancel{(n-1)!}}{n \cdot \underbrace{(n-1)\cdot(n-2)\cdot(n-3)\cdots 3\cdot 2\cdot 1}_{\cancel{(n-1)!}}} = \frac{1}{n}$$

$\frac{a_{n+1}}{a_n} = \frac{1}{n} < 1$ $\qquad$ Decreasing

$a_{n+1} < a_n$

Jul 16-8:34 AM

Find the general term of the Sequence

$$\frac{1}{2}, \frac{-4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \ldots$$

$1^2 \quad 2^2 \quad 3^2 \quad 4^2 \quad 5^2$

$$(-1)^{n+1} \cdot \frac{n^2}{n+1}$$

$$2, \sqrt{2}+\sqrt[3]{2}, \sqrt{3}+\sqrt[3]{3}, \sqrt{4}+\sqrt[3]{4}, \ldots$$

$2 \leftarrow 1+1 \leftarrow \sqrt{1}+\sqrt[3]{1}$

$$\left\{\sqrt{n}+\sqrt[3]{n}\right\}_{n=1}^{\infty}$$

Jul 16-8:44 AM

Now infinite Series (Series)

infinite Series is $\boxed{\text{the Sum of terms}}$ of the corresponding Sequence

$$\{a_n\}_{n=1}^{\infty} \to a_1, a_2, a_3, a_4, \ldots$$

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

Partial Sum $\to \sum_{n=1}^{K} a_n$

ex: $\sum_{n=1}^{4} n^2 = 1^2 + 2^2 + 3^2 + 4^2$

$= 1 + 4 + 9 + 16 = \boxed{30}$

Yesterday $\boxed{1^2+2^2+3^2+\cdots} \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

For $n=4$ $\quad \frac{4(4+1)(2\cdot 4+1)}{6}$

$= \frac{4\cdot 5\cdot 9}{6} = \frac{180}{6} = \boxed{30}$

Jul 16-8:53 AM

Find the partial Sum

$$\sum_{n=1}^{4} n^3 = 1^3 + 2^3 + 3^3 + 4^3 = \boxed{100}$$

From PreCalc.

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

For $n = 4$ $\left[\frac{4(4+1)}{2}\right]^2 = \left[\frac{20}{2}\right]^2 = 10^2 = \boxed{100}$

Jul 16-8:59 AM

Show $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

use mathematical induction

1) Is it true for $n=1$? Yes

LHS 1^3 RHS $\left[\frac{1(1+1)}{2}\right]^2 = 1^2 = 1$

2) what about $n=2$? Yes

LHS $1^3 + 2^3 = 1 + 8 = 9$ RHS $\left[\frac{2(2+1)}{2}\right]^2 = 3^2 = 9$

3) Assume it is true for $n = k$

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$$

Add the next term

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4}$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$= \frac{(k+1)^2[k^2 + 4(k+1)]}{4}$$

$$= \frac{(k+1)^2(k^2+4k+4)}{4} = \frac{(k+1)^2(k+2)^2}{4}$$

$$= \frac{(k+1)^2(k+1+1)^2}{4}$$

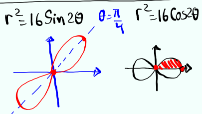
$$= \left[\frac{(k+1)(k+1+1)}{2}\right]^2$$

$$\left[\frac{n(n+1)}{2}\right]^2$$

Correction on the graph of QZ Yesterday

$\theta = \frac{\pi}{4}$, $\theta = \frac{\pi}{2}$

$r^2 = 16\sin 2\theta$ $\theta = \frac{\pi}{4}$ $r^2 = 16\cos 2\theta$



Jul 16-9:03 AM

Consider the Series $\sum_{n=1}^{\infty}\left[\frac{1}{n}-\frac{1}{n+1}\right]$

1) find the $\underbrace{\text{Sum of the first 4 term}}_{S_4}$

$S_4=\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{4}}-\frac{1}{5}\right)$

$n=1$ $\quad n=2$ $\quad n=3$ $\quad n=4$

$=1-\frac{1}{5}=\frac{4}{5}$

S_∞

$\uparrow$

$n\to\infty$

$\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cancel{\cdots}+\left(\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)$

$=1-\cancel{\frac{1}{n+1}}\to 0 \quad =\boxed{1}$

Jul 16-9:32 AM

Consider $\sum_{n=2}^{\infty}\frac{2}{n^2-1}$

$S_4=\left(\frac{2}{2^2-1}\right)+\left(\frac{2}{3^2-1}\right)+\left(\frac{2}{4^2-1}\right)+\left(\frac{2}{5^2-1}\right)$

$=\frac{2}{3}+\frac{2}{8}+\frac{2}{15}+\frac{2}{24}=\boxed{\frac{17}{15}}$

$\frac{2}{n^2-1}=\frac{2}{(n+1)(n-1)}=\frac{A}{n+1}+\frac{B}{n-1}$

$2=A(n-1)+B(n+1)$ $\qquad \frac{2}{n^2-1}=\frac{1}{n-1}-\frac{1}{n+1}$

$n=1$ $\quad 2=A\cdot 0+2B$ $\quad B=1$

$n=-1$ $\quad 2=-2A+B\cdot 0$ $\quad A=-1$

$\sum_{n=2}^{\infty}\frac{2}{n^2-1}=\sum_{n=2}^{\infty}\left[\frac{1}{n-1}-\frac{1}{n+1}\right]$

$S_4=\left(\frac{1}{1}-\cancel{\frac{1}{3}}\right)+\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{3}}-\frac{1}{5}\right)+\left(\cancel{\frac{1}{4}}-\frac{1}{6}\right)$

$n=2$ $\quad n=3$ $\quad n=4$ $\quad n=5$

$=1+\frac{1}{2}-\frac{1}{5}-\frac{1}{6}=\boxed{\frac{17}{15}}$

Jul 16-9:37 AM

Consider the Series below

$a + ar + ar^2 + ar^3 + \cdots\cdots$

First Term

Common ratio

$a + a+d + a+2d + a+3d, \cdots\cdots$

Common difference

Arithmetic & Geometric Series

when each term is multiplied by a Common ratio r

$a, ar, ar^2, \cdots\cdots$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

Infinite geometric Series

Can $r=1$? NO

what if $r>1$? No Sum

there is a Sum only if $|r|<1$

Jul 16-9:45 AM

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{a}{1-r} = \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = \boxed{1}$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\cdots$$

$a = \frac{1}{2}$

$r = \frac{1}{2}$

$$\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n = \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \frac{81}{16} + \cdots$$

$a = \frac{3}{2}$, $r = \frac{3}{2}$

Since $|r| \geq 1$
No sum

Diverges

Jul 16-9:54 AM

Consider $\sum_{n=1}^{\infty} x^n$

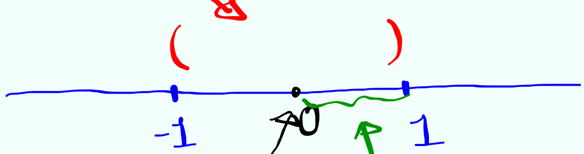
$$= x + x^2 + x^3 + \cdots$$

Interval of Convergence

$a = x$

$r = x$ There is a sum if

$|r| < 1$



$|x| < 1$

$\boxed{-1 < x < 1}$

Center of Convergence

Radius of convergence

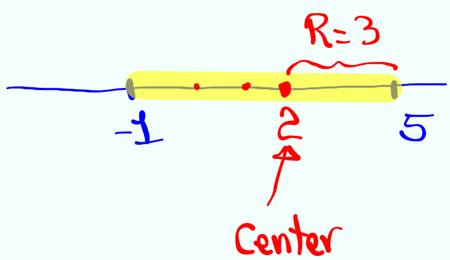
Jul 16-9:57 AM

$$\sum_{n=1}^{\infty} \left(\frac{x-2}{3}\right)^n$$

$a = \frac{x-2}{3}$, $r = \frac{x-2}{3}$

It converges if $|r| < 1$

$$\left|\frac{x-2}{3}\right| < 1$$



$$-1 < \frac{x-2}{3} < 1$$

$$-3 < x-2 < 3$$

$$-1 < x < 5$$

Jul 16-10:01 AM

Find the value of x for which series $\sum_{n=0}^{\infty} \frac{2^n}{x^n}$ converges.

$= 1 + \frac{2}{x} + \frac{4}{x^2} + \frac{8}{x^3} + \cdots$

$a = 1$, $r = \frac{2}{x}$ converges only $|r| < 1$

$\left|\frac{2}{x}\right| < 1$

$\frac{2}{|x|} < 1$

$2 < 1|x|$

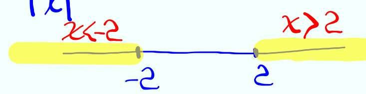
$|x| > 2$

using infinite geometric series formula

$\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r}$

$= \frac{1}{1-\frac{2}{x}} = \boxed{\frac{x}{x-2}}$

simplify



$x = 3$

$S_\infty = \frac{3}{3-2} = \boxed{3}$

Jul 16-10:05 AM

$\sum_{n=0}^{\infty} e^{nx}$ Find values of x for which this converges.

$= e^0 + e^x + e^{2x} + e^{3x} + \cdots$

$a = e^0 = 1$, $r = e^x$ this sum exists only

$|r| < 1$

$|e^x| < 1$

$e^x < 1$

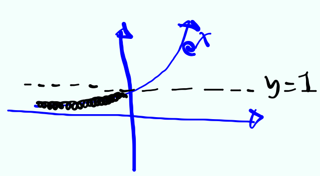
$\ln e^x < \ln 1$

$x \cancel{\ln e} < 0$

$x < 0$

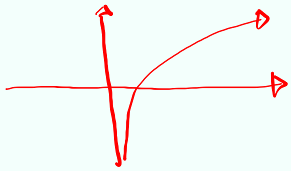
$S_\infty = \frac{a}{1-r} = \boxed{\frac{1}{1-e^x}}$

$x < 0$



Jul 16-10:13 AM

Evaluate $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$



$= \lim_{t\to\infty} \ln x \Big|_1^t$

$= \lim_{t\to\infty} \left(\ln t - \cancelto{0}{\ln 1}\right) = \infty$

Diverges.

$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t x^{-2}\,dx$

$= \lim_{t\to\infty} \left[\frac{x^{-1}}{-1} \Big|_1^t \right] = \lim_{t\to\infty} \frac{-1}{x} \Big|_1^t$

$= \lim_{t\to\infty} \left[\cancelto{0}{\frac{-1}{t}} + \frac{1}{1} \right] = 1$ Converges.

$\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$

Diverges if $p \le 1$

Jul 16-10:21 AM

Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Diverges

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ Converges $p > 1$

$\frac{1}{n} = \frac{1}{n^1}$ $p = 1$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ Diverges $p \le 1$

Jul 16-10:29 AM

Write out the first 3 terms for

$$\left\{ \frac{2^n+3^n}{5^n} \right\}_{n=1}^{\infty}$$

$$\frac{2^1+3^1}{5^1}, \quad \frac{2^2+3^2}{5^2}, \quad \frac{2^3+3^3}{5^3},$$

Find the sum $\sum_{n=1}^{\infty} \frac{2^n+3^n}{5^n}$.

$$\sum_{n=1}^{\infty} \frac{2^n+3^n}{5^n} = \sum_{n=1}^{\infty}\left(\frac{2^n}{5^n}+\frac{3^n}{5^n}\right)$$

$$= \sum_{n=1}^{\infty} \frac{2^n}{5^n} + \sum_{n=1}^{\infty} \frac{3^n}{5^n}$$

$$= \sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n + \sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n$$

$a=\frac{2}{5}$, $r=\frac{2}{5}$, $|r|<1$ Converges

$a=\frac{3}{5}$, $r=\frac{3}{5}$, $|r|<1$ Converges

$$= \frac{\frac{2}{5}}{1-\frac{2}{5}} + \frac{\frac{3}{5}}{1-\frac{3}{5}}$$

$$= \frac{2}{3} + \frac{3}{2} = \boxed{\frac{13}{6}}$$

Jul 16-10:33 AM

Find a reduced fraction that is equal to

$.\overline{3}$ $\qquad \frac{1}{3}$

$N = .\overline{3}$

$\rightarrow N = .33333\ldots$

$10N = 10(.3333\ldots)$

$\rightarrow 10N = 3.33333\ldots$

$10N - N = 3 \qquad 9N = 3 \rightarrow \boxed{N=\frac{1}{3}}$

$.\overline{3} = .3 + .03 + .003 + .0003 + \cdots$

$a=.3 \qquad r=.1 \qquad S_\infty = \frac{a}{1-r} = \frac{.3}{1-.1}$

$$= \frac{.3}{.9} = \frac{1}{3}$$

Jul 16-10:42 AM

Find a reduced fraction that is equal to $.\overline{25}$

$.\overline{25} = .25 + .0025 + .000025 + .00000025 + \ldots$

$a = .25 \qquad r = .01$

$$S_\infty = \frac{a}{1-r} = \frac{.25}{1-.01} = \frac{.25}{.99} = \frac{25}{99}$$

Jul 16-10:47 AM